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Problem of Optimal Pursuit

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Introduction

THE problem of reaching the collinear libration points of the Earth-Moon system of a space vehicle that is placed in Earth or Moon orbit has been discussed in Refs. 1–6.

It has been seen that, using impulsive thrusts,[7] small variations of the initial speed lead to significant deviations from libration points. This disadvantage can be eliminated by the continuous use of a low thrust, which leads to a large class of optimization problems for a variety of performance indices.

The present Note attempts to study the pursuit around the collinear libration points of the Earth-Moon system of two vehicles each having a rocket engine with small acceleration. The optimal thrust acceleration and the corresponding trajectory that allows the interceptor to be situated at a given distance from other vehicles that move on an orbit around the collinear libration points in the Earth-Moon system are determined.

If the transfer time is known, the laws of the optimal motion minimize the consumption of fuel.

Expressions for the controls and optimal state variables are obtained. Also, the minimum consumption of fuel corresponding to the formulated optimal problem is evaluated.

Transfer in a Vacuum

For a space vehicle acted on by a low-thrust propulsion force, the transfer from the libration points to the Earth or Moon orbits is equivalent to a problem of encounter in orbit.

Considering a rotating system having its origin at the collinear libration points for the Earth-Moon system, the linearized system of equations of motion of the space vehicle can be written in the form given in the following equations:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 = f_1\\ \frac{dx_2}{dt} &= K_1x_1 + 2\omega x_4 + u_1 = f_2\\ \frac{dx_3}{dt} &= x_4 = f_3\\ \frac{dx_4}{dt} &= -2\omega x_2 + K_2x_3 + u_2 = f_4\end{aligned} \tag{1}$$

where x_1 and x_3 represent the coordinates of the space vehicle, and x_2 and x_4 are the velocity components; u_1 and u_2 are the thrust acceleration components in the rotating system.

Formulation of the Problem

Let us consider a vehicle that moves on an orbit around the collinear libration points of the Earth-Moon system without being acted on by propulsive force.

Suppose that another intercept vehicle that is situated on a close orbit is endowed with an engine of small propulsion, which causes a variable acceleration. We determine the command laws that transfer the interceptor from the position

$$S_0 = \{x;\ x_i = x_i^0\ (i = 1, \ldots, 4)\} \tag{2}$$

on the terminal surface

$$S = \{x;\ x_1 = s_1,\ x_2 = s_2,\ x_3 = \pm(\delta^2 - s_1^2)^{1/2},\ x_4 = s_4\} \tag{3}$$

such that in a given time T the fuel consumption given by the functional

$$J = \frac{1}{2}\int_0^T (u_1^2 + u_2^2)\,dt \tag{4}$$

is minimal.

Optimum Problem

The index of performance will be considered in its most general form

$$V(x) = \min_{u_k \in U}\left\{\int_t^T G(x,u_k)\,dt + K[x(T)]\right\} \tag{5}$$

From Eq. (4), we note that the index of performance is an integral form; therefore, on the terminal surface we have

$$V(S) = 0 \tag{6}$$

Taking into account Eq. (6), the transversality conditions become

$$\sum_{i=1}^{4}\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial s_j} = 0, \qquad (j = 1, \ldots, 4) \tag{7}$$

From Eq. (7), it follows that

$$V_{x_1}(S) = \frac{V_{x_3}(S)s_1}{\pm(\delta^2 - s_1^2)^{1/2}}, \qquad V_{x_2}(S) = 0, \qquad V_{x_3}(S) = 0 \tag{8}$$

We must determine $u_i = u_i^*$ along the optimal trajectory such that

$$H = \sum_{i=1}^{4} V_{x_i} f_i(x_i u) + \tfrac{1}{2}(u_1^2 + u_2^2) \tag{9}$$

be minimal.

Since u has no constraint concerning its amplitude, the condition of minimal H may be written as

$$\frac{\partial H}{\partial u_k} = 0, \qquad (k = 1,2) \tag{10}$$

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From Eq. (10), we obtain the expression of the command functions

$$u_1^* = -V_{x_2}$$
$$u_2^* = -V_{x_4} \tag{11}$$

An analysis of Eqs. (8) and (11) shows that the modulus of the acceleration vector u is zero on the terminal surface. The adjoint system is written as

$$\frac{dV_{x_i}}{dt} = -\sum_{j=1}^{4} V_{x_j}\frac{\partial f_j}{\partial x_i} \tag{12}$$

By changing the variables

$$\tau = -t + T \tag{13}$$

the system of equations of motion [Eq. (1)] together with the adjoint system in Eq. (12) become

$$\frac{dx_i}{d\tau} = -f_i(x,u) \tag{14a}$$

$$\frac{dV_{x_i}}{d\tau} = \sum_{j=1}^{4} V_{x_j}\frac{\partial f_j}{\partial x_i}, \qquad (i = 1,\dots,4) \tag{14b}$$

Conditions in Eqs. (3) and (8) on the terminal surface where $\tau = 0$ become initial conditions for the differential system [Eq. (14)].

Determination of the Extremals

The adjoint system (14b) is

$$\begin{aligned}\frac{dV_{x_1}}{d\tau} &= K_1V_{x_2}\\ \frac{dV_{x_2}}{d\tau} &= V_{x_1} - 2\omega V_{x_4}\\ \frac{dV_{x_3}}{d\tau} &= K_2V_{x_4}\\ \frac{dV_{x_4}}{d\tau} &= 2\omega V_{x_2} + V_{x_3}\end{aligned} \tag{15}$$

with the characteristic equation

$$\lambda^4 + (4\omega^2 - K_2 - K_1)\lambda^2 + K_1K_2 = 0 \tag{16}$$

which has the solutions

$$x_{1,2} = \pm a, \qquad x_{3,4} = i\beta \tag{17}$$

where

$$a = \left[\frac{K_1 + K_2 - 4\omega^2 + \sqrt{(K_1 + K_2 - 4\omega^2)^2 - 4K_1K_2}}{2}\right]^{1/2}$$
$$\beta = \left[\frac{K_1 + K_2 - 4\omega^2 - \sqrt{(K_1 + K_2 - 4\omega^2)^2 - 4K_1K_2}}{2}\right]^{1/2} \tag{18}$$

Integrating Eqs. (15), we obtain the particular solutions

$$\overline{V}_{x_i} = \begin{pmatrix}\overline{V}_{x_i}^1\\ \overline{V}_{x_i}^2\\ \overline{V}_{x_i}^3\\ \overline{V}_{x_i}^4\end{pmatrix} \tag{19}$$

Taking the fundamental system of solution:

$$V_{x_i} = \overline{V}_{x_1}, \quad V_{x_2} = \overline{V}_{x_2}, \quad V_{x_3} = \frac{\overline{V}_{x_3} + \overline{V}_{x_4}}{2}, \quad V_{x_4} = \frac{\overline{V}_{x_3} - \overline{V}_{x_4}}{2i} \tag{20}$$

we finally obtain

$$V_{x_1} = C_1a(4\omega^2 - K_2 + a^2)e^{a\tau} - C_2a(4\omega^2 - K_2 + a^2)e^{-a\tau}$$
$$- C_3b(4\omega^2 - K_2 - \beta^2)\sin\beta\tau + C_4b(4\omega^2 - K_2 - \beta^2)\cos\beta\tau \tag{21a}$$

$$V_{x_2} = C_1(a^2 - K_2)e^{a\tau} + C_2(a^2 - K_2)e^{-a\tau} - C_3(b^2 + K_2)\cos\beta\tau$$
$$- C_4(b^2 + K_2)\sin\beta\tau \tag{21b}$$

$$V_{x_3} = C_1(2\omega K_2)e^{a\tau} + C_2(2\omega K_2)e^{-a\tau} + C_3(2\omega K_2)\cos\beta\tau$$
$$+ C_4(2\omega K_2)\sin\beta\tau \tag{21c}$$

$$V_{x_4} = C_1(2\omega a)e^{a\tau} - C_2(2\omega a)e^{-a\tau} - C_3(2\omega\beta)\sin\beta\tau$$
$$+ C_4(2\omega\beta)\cos\beta\tau \tag{21d}$$

The adjoint system has the same characteristic equation as the homogeneous ($u_K = 0$) system of the equations of motion in the variable, and the solution of this last one may be written similarly with

$$\overline{x}_1 = \overline{C}_1a(4\omega^2 - K_2 + a^2)e^{a\tau} - \overline{C}_2a(4\omega^2 - K_2 + a^2)e^{-a\tau}$$
$$- \overline{C}_3\beta(4\omega^2 - K_2 + \beta^2)\sin\beta\tau + \overline{C}_4\beta(4\omega^2 - K_2 - \beta^2)\cos\beta\tau \tag{22a}$$

$$\overline{x}_2 = \overline{C}_1(a^2 - K_2)e^{a\tau} + \overline{C}_2(a^2 - K_2)e^{-a\tau} - \overline{C}_3(\beta^2 + K_2)\cos\beta\tau$$
$$- \overline{C}_4(\beta^2 + K_2)\sin\beta\tau \tag{22b}$$

$$\overline{x}_3 = \overline{C}_1 2\omega K_1e^{a\tau} + \overline{C}_2 2\omega K_1e^{-a\tau} + \overline{C}_3 2\omega K_1\cos\beta\tau$$
$$+ \overline{C}_4 2\omega K_1\sin\beta\tau \tag{22c}$$

$$\overline{x}_4 = -\overline{C}_1 2\omega K_1ae^{a\tau} + \overline{C}_2 2\omega K_1ae^{a\tau} + \overline{C}_3 2\omega K_1\beta\sin\beta\tau$$
$$- \overline{C}_4 2\omega K_1\beta\cos\beta\tau \tag{22d}$$

A particular solution of the nonhomogeneous system is obtained by using the variation of constants. We have

$$\overline{C}_1(\tau) = \frac{a^2 + \beta^2}{4a\omega K_1[4\omega^2(K_2 - a^2) + 2a^2(\beta^2 + K_2) + \beta^4 - K_2^2]}$$
$$\times \Bigg\{C_1(2\omega a)(1 - a^2 + K_2)\tau + C_2\omega(1 + a^2 - K_2)e^{-2a\tau}$$
$$- C_3 2\omega\left[-\frac{\beta}{a^2 + \beta^2}(ae^{-a\tau}\sin\beta\tau + \beta e^{-a\tau}\cos\beta\tau)\right.$$
$$\left.- \frac{a(\beta^2 + K_2)}{a^2 + \beta^2}(ae^{-a\tau}\sin\beta\tau + \beta e^{-a\tau}\cos\beta\tau)\right]$$
$$+ C_4\left[\frac{2\omega\beta}{a^2 + \beta^2}(\beta e^{-a\tau}\sin\beta\tau - ae^{-a\tau}\cos\beta\tau)\right]\Bigg\} \tag{23a}$$

$$\overline{C}_2(\tau) = \frac{1}{2aK_1[4\omega^2(K_2 - a^2) + 2a^2(\beta^2 + K_2) + \beta^4 - K_2^2]}$$
$$\times \Big\{ \tfrac{1}{2}\,[2a^2(\beta^2 + K_2) + (a^2 + \beta^2)(\beta^2 - a^2 - 4\omega^2)]\,C_1$$
$$+ a[4\omega^2(a^2 + \beta^2) - 2\beta^2 K_2 - (a^4 + \beta^4)]\,C_2\tau$$
$$+ \left[-\frac{a(\beta^2 - K_2)(\beta^2 + K_2)}{a^2 + \beta^2}(\beta e^{a\tau}\sin\beta\tau + ae^{a\tau}\cos\beta\tau)\right.$$
$$\left. + \beta(4\omega^2 - K_2 - \beta^2)(ae^{a\tau}\sin\beta\tau - \beta e^{a\tau}\cos\beta\tau)\right] C_3$$
$$- \left[\frac{a(\beta^2 - a^2)(\beta^2 + K_2)}{a^2 + \beta^2}(ae^{a\tau}\sin\beta\tau - \beta e^{a\tau}\cos\beta\tau)\right.$$
$$\left. + \beta(4\omega^2 - K_2 - \beta^2)(\beta e^{a\tau}\sin\beta\tau + ae^{a\tau}\cos\beta\tau)\right] C_4 \Big\} \tag{23b}$$

$$\overline{C}_3(\tau) = \frac{a^2 + \beta^2}{\beta K_1[4\omega^2(K_2 - a^2) + 2a^2(\beta^2 + K_2) + \beta^4 - K_2^2]}$$
$$\times \Big\{ \left[\frac{\beta(a^2 - K_2)}{a^2 + \beta^2}(\beta e^{a\tau}\sin\beta\tau + ae^{a\tau}\cos\beta\tau)\right.$$
$$\left. + \frac{a(4\omega^2 - K_2 + a^2)}{a^2 + \beta^2}(ae^{a\tau}\sin\beta\tau - \beta e^{a\tau}\cos\beta\tau)\right] C_1$$
$$+ \left[\frac{\beta(a^2 - K_2)}{a^2 + \beta^2}(\beta e^{-a\tau}\sin\beta\tau - ae^{-a\tau}\cos\beta\tau)\right.$$
$$\left. + \frac{a(4\omega^2 - K_2 + a^2)}{a^2 + \beta^2}(ae^{-a\tau}\sin\beta\tau + \beta e^{-a\tau}\cos\beta\tau)\right] C_2$$
$$+ \left[-\frac{\beta(\beta^2 + K_2)}{2}\left(\tau + \frac{1}{\beta}\sin 2\beta\tau\right) - \frac{\beta(4\omega^2 - K_2 + a^2)}{2}\right.$$
$$\left. \times \left(\tau - \frac{1}{\beta}\sin 2\beta\tau\right)\right] C_3 + \frac{4\omega^2 - 2K_2 + a^2 - \beta^2}{4}\cos 2\beta\tau\, C_4 \Big\} \tag{23c}$$

$$\overline{C}_4(\tau) = \frac{a^2 + \beta^2}{\beta K_1[4\omega^2(K_2 - a^2) + 2a^2(\beta^2 + K_2) + \beta^4 - K_2^2]}$$
$$\times \Big\{ \left[\frac{\beta(a^2 - K_2)}{a^2 + \beta^2}(ae^{a\tau}\sin\beta\tau - \beta e^{a\tau}\cos\beta\tau)\right.$$
$$\left. - \frac{2\omega a(4\omega^2 - K_2 + a^2)}{a^2 + \beta^2}(\beta e^{a\tau}\sin\beta\tau + ae^{a\tau}\cos\beta\tau)\right] C_1$$
$$+ \left[-\frac{\beta(a^2 - K_2)}{a^2 + \beta^2}(ae^{-a\tau}\sin\beta\tau + \beta e^{-a\tau}\cos\beta\tau)\right.$$
$$\left. + \frac{2\omega a(4\omega^2 - K_2 + a^2)}{a^2 + \beta^2}(\beta e^{-a\tau}\sin\beta\tau - ae^{-a\tau}\cos\beta\tau)\right] C_2$$
$$- \frac{1}{4}\left[-(\beta^2 + K_2) + 2\omega(4\omega^2 - K_2 + a^2)\right]\cos 2\beta\tau\, C_3$$
$$+ \left[-\frac{\beta(\beta^2 + K_2)}{2}\left(\tau - \frac{1}{\beta}\sin 2\beta\tau\right) - \cos\beta(4\omega^2 - K_2 + a^2)\right.$$
$$\left. \times \left(\tau + \frac{1}{\beta}\sin 2\beta\tau\right)\right] C_4 \Big\} \tag{23d}$$

Replacing $C_i(\tau)$ in Eqs. (22), we obtain the particular solution x_i^b for the nonhomogeneous system. Splitting the solution of the system of the equations of motions as

$$x_i = \bar{x}_i + x_i^b \tag{24}$$

we obtain the expressions of the state variables on the optimal trajectory

$$x_i(\tau) = \alpha_{i1}e^{a\tau} + \alpha_{i2}e^{-a\tau} + \alpha_{i3}\tau e^{a\tau} + \alpha_{i4}\tau e^{-a\tau} + \alpha_{i5}\sin\beta\tau$$
$$+ \alpha_{i6}\cos\beta\tau + \alpha_{i7}\tau\sin\beta\tau + \alpha_{i8}\tau\cos\beta\tau + \beta_{i1}e^{a\tau}\cos 2\beta\tau$$
$$+ \beta_{i2}e^{a\tau}\sin 2\beta\tau + \beta_{i3}e^{-a\tau}\cos 2\beta\tau + \beta_{i4}e^{-a\tau}\sin 2\beta\tau$$
$$+ \beta_{i5}\sin\beta\tau\sin 2\beta\tau + \beta_{i6}\sin 2\beta\tau\cos 2\beta\tau$$
$$+ \beta_{i7}\cos\beta\tau\cos 2\beta\tau + \beta_{i8}\cos\beta\tau\sin 2\beta\tau \tag{25}$$

where α_{ik} and β_{ik} ($i = 1,\ldots,4$; $k = 1,\ldots,8$) are constants resulting from the computation.

The conditions $x_i(\tau) \in S$ may be written as

$$x_i(0) = s_i, \quad (i = 1,2,4); \qquad x_3(0) = \pm(\delta^2 - s_1^2)^{1/2} \tag{26}$$

or, explicitly,

$$\begin{aligned} a(4\omega^2 - K_2 + a^2)(\overline{C}_1 - \overline{C}_2) + \beta(4\omega^2 - K_2 - \beta^2)\overline{C}_4 &= s_1 + \alpha_1 \\ -K_1(a^2 - K_2)(\overline{C}_1 + \overline{C}_2) + K_1(\beta^2 + K_2)\overline{C}_3 &= s_2 + \alpha_2 \\ 2\omega K_1(\overline{C}_1 + \overline{C}_2 + \overline{C}_3) &= \pm(\delta^2 - s_1^2)^{1/2} + \alpha_3 \\ -2\omega K_1 a(\overline{C}_1 - \overline{C}_2) - 2\omega K_1\beta\overline{C}_4 &= s_4 + \alpha_4 \end{aligned} \tag{27}$$

From Eq. (27), it follows that

$$\begin{aligned} \overline{C}_1 &= \frac{2\omega K_1(s_1 + \alpha_1) - 2\omega a(s_2 + \alpha_2) + a(\beta^2 + K_2)[\pm(\delta^2 - s_1^2)^{1/2} + \alpha_3] + (4\omega^2 - K_2 + \beta^2)(s_4 + \alpha_4)}{4\omega K_1 a(a^2 + \beta^2)} \\ \overline{C}_2 &= \frac{-2\omega K_1(s_1 + \alpha_1) - 2\omega a(s_2 + \alpha_2) + a(\beta^2 + K_2)[\pm(\delta^2 - s_1^2)^{1/2} + \alpha_3] - (4\omega^2 - K_2 - \beta^2)(s_4 + \alpha_4)}{4\omega K_1 a(a^2 + \beta^2)} \\ \overline{C}_3 &= \frac{2\omega(s_2 + \alpha_2) + (a^2 - K_2)[\pm(\delta^2 - s_1^2) + \alpha_3]}{2\omega K_1(a^2 + \beta^2)} \\ \overline{C}_4 &= -\frac{2\omega K_1(s_1 + \alpha_1) + (4\omega^2 - K_2 + a^2)(s_4 + \alpha_4)}{2\omega K_1\beta(a^2 + \beta^2)} \end{aligned} \tag{28}$$

Taking into account the expressions of the adjoint variables using conditions [Eq. (8)], we obtain the system

$$
\begin{aligned}
&C_1(a^2 - K_2) + C_2(a^2 - K_2) - C_3(\ell^2 + K_2) = 0 \\
&C_1(2\omega a) - C_2(2\omega a) + C_4(2\omega \ell) = 0 \\
&C_1 a(4\omega^2 - K_2 + a^2) - C_2 a(4\omega^2 - K_2 + a^2) \\
&\quad + C_4 \ell(4\omega^2 - K_2 - \ell^2) = \frac{2\omega K_2(C_1 + C_2 + C_3)}{\pm(\delta^2 - s_1^2)}
\end{aligned}
\tag{29}
$$

Owing to the linearity of the constants C_i ($i = 1, \ldots, 4$), we may consider only three arbitrary constants, for instance, C_2/C_1, C_3/C_1, and C_4/C_1. Thus, we obtain

$$
\begin{aligned}
\frac{C_2}{C_1} &= \frac{\pm a(\ell^2 + K_2)(\delta^2 - s_1^2)^{1/2} - 2\omega K_2 s_1}{\pm a(\ell^2 + K_2)(\delta^2 - s_1^2) + 2\omega K_2 s_1} \\
\frac{C_3}{C_1} &= \frac{\pm 2a(a^2 - K_2)(\delta^2 - s_1^2)^{1/2}}{\pm a(\ell^2 + K_2)(\delta^2 - s_1^2)^{1/2} + 2\omega K_2 s_1} \\
\frac{C_4}{C_1} &= \frac{-4a\omega K_2 s_1}{\ell\left[\pm(\ell^2 + K_2)(\delta^2 - s_1^2)^{1/2} + 2\omega K_2 s_1\right]}
\end{aligned}
\tag{30}
$$

A restricted family of trajectories may be obtained by taking, for instance, $C_2/C_1 = 0$. By analyzing the expressions of the constants α_{ij} and β_{ij} ($i, j = 1, \ldots, 8$), we note that they depend on C_1 and the parameters s_1, s_2, and s_4. These may be determined from the condition $x_i(\tau) \in S^o$, which reads

$$x_i(T) = x_i^0, \qquad (i = 1, \ldots, 4) \tag{31}$$

Using Eqs. (15) and (20), the expression of the minimal fuel consumption becomes

$$
\begin{aligned}
V &= \gamma_1 + \gamma_2 T + \gamma_3 e^{2aT} + \gamma_4 e^{-2aT} + \gamma_5 \sin 2\ell T \\
&\quad + \gamma_6 \cos 2\ell T + \gamma_7 e^{aT} \sin \ell T + \gamma_8 e^{aT} \cos \ell T \\
&\quad + \gamma_9 e^{-aT} \sin \ell T + \tau_{10} e^{-aT} \cos \ell \tau
\end{aligned}
\tag{32}
$$

Conclusions

In this analysis, we determined the optimal control of the acceleration vector and the trajectory of the interceptor such that the fuel consumption necessary to the maneuver of interception be minimal.

The acceleration program and the corresponding trajectory have been obtained in a closed form. The problem of obtaining the integration constants that appear in the equations requires solution of nonlinear algebraic equations.

The present Note furnishes the analytical formulas for the characteristics of the optimal motion in terms of the parameters of the terminal surface.

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Point Stacking Technique for Set Matching

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Introduction

TWO sets of vectors A and B are equal (approximately equal) through translation if a vector c can be found such that A and the translate $B + c$ are equal (approximately equal) as point sets ($B + c$ = translate of $B = \{b + c: b$ is a member of $B\}$). In applications, A is often provided as a table or catalog of known vectors, B is a collection of measurements of the vectors in A, and c is the unknown translational vector to be estimated. As pointed out in Ref. 1, false observations in B can lead to erroneous estimates c and result in false matching between vectors in the two sets. In this Note, a simple, dependable procedure called "point stacking" is presented that is not sensitive to false observations in B, provided that a minimal amount of valid B data are available. The method finds useful applications in angular separation matching, phase matching (see Ref. 1), Earth point-source recognition problems, and general n-dimensional set matching problems. As a graphical tool, point stacking provides fast visual solutions to simple set matching problems.

Stacking Method Theory

Consider two sets of constant, known n-dimensional vectors $A = \{a(1), a(2), \ldots, a(N)\}$ and $B = \{b(1), b(2), \ldots, b(M)\}$. Suppose the latter set defines a fixed translation of a particular subset of the former set; i.e., there exists an unknown n-dimensional vector c such that $b(j) + c$ is a member of $\{a(1), \ldots, a(N)\}$ for each $j = 1, \ldots, M$. The aim of point stacking is the estimation of the translational vector c, as well as the vectors in A that best correspond to the vectors in B through translation by c.

To this end, consider the union S defined as

$$
\begin{aligned}
S &= \bigcup_{j=1}^{M} A - [b(j) - x] \\
&= \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} [a(i) - b(j) + x]
\end{aligned}
\tag{1}
$$

where x is an arbitrarily chosen, fixed n-dimensional vector (in most applications $x = 0$ is appropriate). Thus defined, S consists of M translates of the set A, which amounts to the collection of all expressions $a(i) - b(j) + x$, $i \leq N$, $j \leq M$. Since by assumption $b(j) + c$ is a member of A for each $j = 1, \ldots, M$, it follows that $x + c$ is a member of the translate $A - [b(j) - x]$ for $j = 1, \ldots, M$. In other words, $x + c$ appears exactly M times in the union S in Eq. (1). Assuming that

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